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LETTER TO THE EDITOR

Operator identities, representations of algebras and the problem of normal ordering

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Abstract. Families of operator identities related to certain powers of positive root generators of (super) Lie algebras of first-order differential operators and q-deformed algebras of first-order finite-difference operators are presented. It is shown that those identities once rewritten in terms of creation/annihilation operators lead to a simplification of the problem of the normal ordering in the second quantization method.

The method of the second quantization is one of the main tools in quantum field theory and statistical mechanics. One of the tedious problems appearing in applications of this method is the problem of normal ordering. This letter is devoted to a description of certain infinite families of relations between creation/annihilation operators, which can simplify the problem of the normal ordering. Those relations occur as a consequence of the existence of finite-dimensional representations of semi-simple Lie algebras.

The following *operator identity* holds

$$(J_n^+)^{n+1} \equiv (x^2\partial_x - nx)^{n+1} = x^{2n+2}\partial_x^{n+1} \qquad \partial_x \equiv \frac{\mathrm{d}}{\mathrm{d}x} \qquad n = 0, 1, 2, \ldots . \tag{1}$$

The proof is straightforward:

(i) the operator $(J_n^+)^{n+1}$ annihilates the space of all polynomials of degree not higher than n, $\mathcal{P}_n(x) = \mathrm{Span}\{x^i : 0 \leqslant i \leqslant n\}$;

(ii) in general, an $(n+1)$th-order linear differential operator annihilating $\mathcal{P}_n(x)$ must have the form $B(x)\partial_x^{n+1}$, where $B(x)$ is an arbitrary function and

(iii) since $(J_n^+)^{n+1}$ is a graded operator, $\deg(J_n^+) = +1$§, $\deg(J_n^+)^{n+1} = n+1$, hence $B(x) = bx^{2n+2}$ while clearly the constant $b = 1$.

It is worth noting that taking the degree in (1) different from $(n+1)$, the left-hand side in (1) will contain immediately all derivative terms from zero up to $(n+1)$th order.

The identity has a Lie-algebraic interpretation. The operator (J_n^+) is the positive-root generator of the algebra sl_2 of first-order differential operators (the other sl_2-generators are

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§ So J_n^+ maps x^k to a multiple of x^{k+1}.

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$J_n^0 = x\partial_x - n/2$, $J_n^- = \partial_x$). Correspondingly, the space $\mathcal{P}_n(x)$ is nothing but the $(n+1)$-dimensional irreducible representation of sl_2. The identity (1) is a consequence of the fact that $(J_n^+)^{n+1} = 0$ in matrix representation.

Another Lie-algebraic interpretation of (1) is connected with occurrence of some relations between the elements of the universal enveloping algebra of the one-dimensional Heisenberg algebra $\{P, Q, 1\}$. Once $[P, Q] = 1$, then:

$$(Q^2P - nQ)^{n+1} = Q^{2n+2}P^{n+1} \qquad n = 0, 1, 2, \ldots. \tag{2}$$

Now let us introduce generators $a = 2^{1/2}P$ and $a^+ = 2^{1/2}Q$. Then (2) takes the form

$$(a^+a^+a - 2na^+)^{n+1} = (a^+)^{2n+2}a^{n+1} \qquad n = 0, 1, 2, \ldots. \tag{3}$$

Definitely, one can interpret the operators a^+, a as creation/annihilation operators, respectively. One can name (3) the *first ordering formula*. Of course, those operators can be realized in the standard way: $a^+ = \partial_x + x$ and $a = \partial_x - x$.

There exist other algebras of differential or finite-difference operators (in more than one variable), which admit a finite-dimensional representation. This leads to more general and *remarkable operator identities and hence to ordering formulae.*

The Lie-algebraic interpretation presented above allows us to generalize (1) for the case of differential operators of several variables, taking appropriate degrees of the highest-positive-root generators of (super) Lie algebras of first-order differential operators, possessing a finite-dimensional invariant sub-space (see e.g. [1]). First we consider the case of sl_3. There exists a representation of $sl_3(\mathbf{C})$ as differential operators on $\mathbf{C}^2$. One of the generators is

$$J_2^1(n) = x^2\partial_x + xy\partial_y - nx.$$

The space $\mathcal{P}_n(x, y) = \mathrm{Span}\{x^iy^j : 0 \leqslant i + j \leqslant n\}$ is a finite-dimensional representation for sl_3, and due to the fact $(J_2^1(n))^{n+1} = 0$ on the space $\mathcal{P}_n(x, y)$, hence we arrive at

$$(J_2^1(n))^{n+1} = (x^2\partial_x + xy\partial_y - nx)^{n+1} = \sum_{k=0}^{k=n+1} \binom{n+1}{k} x^{2n+2-k}y^k\partial_x^{n+1-k}\partial_y^k. \tag{4}$$

This identity is valid for $y \in \mathbf{C}$ (as described above), but also if y is a Grassmann variable, i.e. $y^2 = 0$†. In the last case, $J_2^1(n)$ is a generator of $osp(2, 2)$, see [1].

In general taking sl_k instead of sl_3, the following operator identity holds

$$\begin{aligned}(J_{k-1}^{k-2}(n))^{n+1} &\equiv \left(x_1\sum_{m=1}^{k}(x_m\partial_{x_m} - n)\right)^{n+1} \\ &= x_1^{n+1} \sum_{j_1+j_2+\ldots+j_k=n+1} C^{n+1}_{j_1,j_2,\ldots,j_k} x_1^{j_1}x_2^{j_2}\ldots x_k^{j_k}\partial_{x_1}^{j_1}\partial_{x_2}^{j_2}\ldots\partial_{x_k}^{j_k}\end{aligned} \tag{5}$$

where $C^{n+1}_{j_1,j_2,\ldots,j_k}$ are the generalized binomial (multinomial) coefficients. If $x \in \mathbf{C}^k$ then $J_{k-1}^{k-2}(n)$ is a generator of the algebra $sl_{k+1}(\mathbf{C})$ [1], while some of the variables x's are

† In this case just two terms in the left-hand side of (4) survive.

Grassmann ones, the operator $J_{k-1}^{k-2}(n)$ is a generator of a certain super Lie algebra of first-order differential operators. The operator on the left-hand side of (5) annihilates the linear space of polynomials $\mathcal{P}_n(x_1, x_2, \ldots, x_k) = \mathrm{Span}\{x_1^{j_1} x_2^{j_2} \ldots x_k^{j_k} : 0 \leqslant j_1 + j_2 + \ldots + j_k \leqslant n\}$.

Denoting $Q_m = x_m$ and $P_m = \partial_{x_m}$, one can make the following statement. Once the operators Q_m, P_m are the generators of k-dimensional Heisenberg algebra:

$$[P_m, Q_l] = \delta_{ml}$$

then

$$\left(Q_1 \sum_{m=1}^{k} (Q_m P_m - n)\right)^{n+1} = Q_1^{n+1} \sum_{j_1+j_2+\ldots+j_k=n+1} C_{j_1,j_2,\ldots,j_k}^{n+1} Q_1^{j_1} Q_2^{j_2} \ldots Q_k^{j_k} P_1^{j_1} P_2^{j_2} \ldots P_k^{j_k} \tag{6}$$

(cf (2)). Introducing new operators $a_m = 2^{1/2} P_m$ and $a_m^+ = 2^{1/2} Q_m$, we arrive at

$$\left(a_1^+ \sum_{m=1}^{k} (a_m^+ a_m - 2n)\right)^{n+1} = (a_1^+)^{n+1} \sum_{j_1+j_2+\ldots+j_k=n+1} C_{j_1,j_2,\ldots,j_k}^{n+1} (a_1^+)^{j_1} (a_2^+)^{j_2} \ldots (a_k^+)^{j_k} a_1^{j_1} a_2^{j_2} \ldots a_k^{j_k} \tag{7}$$

(cf (3)). As before, one can consider a standard representation of the operators $a_k^+ = \partial_{x_k} + x_k$, $a_k = \partial_{x_k} - x_k$ as creation/annihilation operators, respectively. One can name (7) the *kth ordering formula*.

The above-described family of operator identities (1) can be generalized for the case of finite-difference operators with the Jackson symbol, D_x (see e.g. [2])

$$D_x f(x) = \frac{f(x) - f(q^2 x)}{(1 - q^2)x} + f(q^2 x) D_x$$

instead of the ordinary derivative. Here, q is an arbitrary complex number. The following operator identity holds

$$(\tilde{J}_n^+)^{n+1} \equiv (x^2 D_x - \{n\} x)^{n+1} = q^{2n(n+1)} x^{2n+2} D_x^{n+1} \qquad n = 0, 1, 2, \ldots \tag{8}$$

(cf (1)), where $\{n\} = (1 - q^{2n})/(1 - q^2)$ is the so-called q-number. The operator in the right-hand side annihilates the space $\mathcal{P}_n(x)$. The proof is similar to the proof of the identity (1).

From an algebraic point of view the operator $\tilde{J}_n^+$ is the generator of a q-deformed algebra $sl_2(\mathbf{C})_q$ of first-order finite-difference operators on the line: $\tilde{J}_n^0 = xD - \hat{n}$, $\tilde{J}_n^- = D$, where $\hat{n} \equiv \{n\}\{n+1\}/\{2n+2\}$ (see [3] and also [1]), obeying the commutation relations

$$\begin{aligned} q^2 \tilde{j}^0 \tilde{j}^- - \tilde{j}^- \tilde{j}^0 &= -\tilde{j}^- \\ q^4 \tilde{j}^+ \tilde{j}^- - \tilde{j}^- \tilde{j}^+ &= -(q^2 + 1) \tilde{j}^0 \\ \tilde{j}^0 \tilde{j}^+ - q^2 \tilde{j}^+ \tilde{j}^0 &= \tilde{j}^+ \end{aligned} \tag{9}$$

($\tilde{j}$'s are related to $\tilde{J}$'s through some multiplicative factors). The algebra (9) has the linear space $\mathcal{P}_n(x)$ as a finite-dimensional representation.

Evidently, the identity (8) has a more general meaning like the identity (1). Once two operators $\tilde{P}$, $\tilde{Q}$ obey a condition $\tilde{P}\tilde{Q} - q^2\tilde{Q}\tilde{P} = 1$, then

$$(\tilde{Q}^2\tilde{P} - \{n\}\tilde{Q})^{n+1} = q^{2n(n+1)}\tilde{Q}^{2n+2}\tilde{P}^{n+1} \qquad n = 0, 1, 2, \ldots \tag{10}$$

(cf (2)).

An attempt to generalize (4), replacing continuous derivatives by Jackson symbols, immediately leads to the necessity to introduce the quantum plane and q-differential calculus [4]

$$\begin{aligned} &xy = qyx \\ &D_x x = 1 + q^2 x D_x + (q^2 - 1) y D_y \qquad D_x y = q y D_x \\ &D_y x = q x D_y \qquad D_y y = 1 + q^2 y D_y \\ &D_x D_y = q^{-1} D_y D_x . \end{aligned} \tag{11}$$

The formulae analogous to (4) have the form

$$\begin{aligned} (\tilde{J}_2^1(n))^{n+1} &\equiv (x^2 D_x + xyD_y - \{n\}x)^{n+1} \\ &= \sum_{k=0}^{k=n+1} q^{2n^2-n(k-2)+k(k-1)} \binom{n+1}{k}_q x^{2n+2-k} y^k D_x^{n+1-k} D_y^k \end{aligned} \tag{12}$$

where

$$\binom{n}{k}_q \equiv \frac{\{n\}!}{\{k\}!\{n-k\}!} \qquad \{n\}! = \{1\}\{2\}\ldots\{n\}$$

are the q-binomial coefficient and q-factorial, respectively. Like all previous cases, if $y \in \mathbf{C}$, the operator $\tilde{J}_2^1(n)$ is one of generators of q-deformed algebra $sl_3(\mathbf{C})_q$ of finite-difference operators, acting on the quantum plane and having the linear space $\mathcal{P}_n(x, y) = \mathrm{Span}\{x^i y^j : 0 \leqslant i + j \leqslant n\}$ as a finite-dimensional representation; the left-hand side of (12) annihilates $\mathcal{P}_n(x, y)$. If y is Grassmann variable, $\tilde{J}_2^1(n)$ is a generator of the q-deformed superalgebra $osp(2, 2)_q$ possessing finite-dimensional representation (see e.g. [1]).

As has been done before (see (2), (3), (6), (7), (10)), the identity (12) can be rewritten in an abstract form replacing x, y and D_x, D_y by abstract operators obeying relations (11).

Introducing a quantum hyperplane [4], one can generalize the whole family of the operator identities (5)–(6) replacing continuous derivatives by finite-difference operators and then by abstract operators, obeying a certain q-deformed Heisenberg algebra.

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